LOCAL INTERPOLATION AND INTERPOLATING BASES

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ABSTRACT. A Schauder basis was found in [5] for the space of Whitney functions given on the Cantor-type set. Here we show that the system suggested corresponds to local interpolations of functions. The same method gives a topological bases in the spaces of continuous functions defined on compact sets of the Cantor type. Elements of the basis locally are polynomials of any preassigned degree.

1. Introduction

Faber [3] found in 1910 the basis in the space $C[0,1]$ consisting of the primitives of the Haar functions. In 1927 the more general form of the result was rediscovered by Schauder [18], who considered not only the diadic system of points. The Faber-Schauder system is interpolating, that is the $n-$th partial sum of the basis expansion of a function f gives the values of f at the first n points from the sequence fixed beforehand. In 1966 V.I. Gurariĭ [8] showed that for any dense sequence (x_n) in a metric compact set K there exists a basis in the space $C(K)$ which is interpolating with nodes (x_n). V.D. Milman ([13], p.119) illustrated this approach by constructing a basis from broken lines in the space of continuous functions defined on the classical Cantor set. The review of up-to-date results on interpolating bases in spaces of continuous functions can be found in [19], Ch.1.3. In particular, the Haar system is considered there (Prop. 2.2.5) as the simplest topological basis in the space $C(K)$ for the Cantor set K. A diadic interpolating basis (P_n) for $C[0,1]$ with moderate growth of degrees of the polynomials P_n was done in [1]. Grober and Bychkov showed in [7] that the Schauder system is an interpolating basis in the space of Hölder type functions $C_\alpha[0,1]$. For interpolating Schauder bases in functional spaces on fractal sets see [10] and [11]. Nevertheless not all functional spaces possess interpolating bases ([9]).

Here we use local interpolations to construct a topological basis in the spaces of continuous functions on Cantor-type sets. Elements of the basis locally are polynomials of any degree given beforehand. The approach allows to find a basis in Banach spaces of differentiable functions defined on rarefied sets (in preparation).

A polynomial basis (P_n) in a functional space is called a Faber basis if $deg\, P_n = n$ for all n. Due to the classical result of Faber [4], the space $C[0,1]$ does not possess a such basis. It should be noted that due to Al.A. Privalov [17] for any $\varepsilon > 0$ in the space $C[0,1]$ there exists a polynomial basis (P_n) with $deg\, P_n \leq (1+\varepsilon)n$.

On the other hand Obermaier found in [15] (see also [16]) a Faber interpolating basis in the space $C(K)$ for $K = \{0\} \cup (q^n)_{n=0}^\infty, 0 < q < 1$. Contrary to the case of $C^\infty-$ functions, where the space $\mathcal{E}(K)$ has an interpolating Faber basis if the Cantor-type

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set is polar ([5]), here we cannot suggest a Faber basis even for very thin Cantor sets. One may ask
Does there exist a perfect set K such that the space $C(K)$ possesses a Faber basis?
More generally this is a problem of a geometric characterization of compact sets such that the corresponding space of continuous functions has a Faber basis. The problem is intimately related to the question of the order of growth of the sequence of the Lebesgue constants for the corresponding compact set.

The choice of nodes is of principal importance in interpolation. For Cantor-type sets we use the *rule of increase of the type*. Locally the points satisfy the Leja condition, that is every next point furnishes the maximum modulus on the corresponding part of the set for the polynomial defined by the previous points as its zeros. But the sequence suggested is not a Leja sequence even for very thin Cantor sets ([6]).

2. **Local Interpolation**

Suppose for infinite compact sets K_0, K_1 we have $K_1 \subset K_0$ and $K_0 \setminus K_1$ is closed. Let natural numbers N_0, M_1, N_1 be given with $N_0 \geq 2$, $M_1 \leq N_0, M_1 \leq N_1$. Let for $s \in \{0,1\}$ we have a finite system of points $(x_k^{(s)})_{k=1}^{N_s} \subset K_s$. Here we suppose that $x_k^{(s)} \neq x_l^{(s)}$ for $k \neq l$ and $(x_k^{(0)})_{k=1}^{N_0-M_1} \subset K_0 \setminus K_1$, $x_{N_0-M_1+r}^{(0)} = x_r^{(1)}$ for $r = 1, \cdots, M_1$. We adopt the conventions that $\sum_{k=m}^{n}(\cdots) = 0$ and $\prod_{k=m}^{n}(\cdots) = 1$ for $m > n$. For $s \in \{0,1\}$, $0 \leq n \leq N_s$ set $\tilde{e}_{ns}(x) = \prod_{k=1}^{n}(x - x_k^{(s)})$, and let e_{ns} be the restriction of $\tilde{e}_{ns}$ to K_s, otherwise $e_{ns}(x) = 0$. Also for any function f defined on K_s let $\xi_{ns}(f) = [x_1^{(s)}, x_2^{(s)}, \cdots, x_{n+1}^{(s)}]f$, where $x_{N_0+1}^{(0)} := x_{M_1+1}^{(1)}$ and $x_{N_1+1}^{(1)} \in K_1$ is any point differing from $x_k^{(1)}$, $k = 1, \cdots, N_1$. For the definition and the properties of the divided differences, see e.g. [2].

Clearly, the system $(e_{ns}, \xi_{ns})_{n=0}^{N_s}$ is biorthogonal, that is $\xi_{ns}(e_{ms}) = \delta_{mn}$. As in [5] we use the functionals

$$\eta_{n,1} = \xi_{n,1} - \sum_{k=n}^{N_0} \xi_{n,1}(e_{k0})\, \xi_{k0}.$$

Only the values $n = M_1 + 1, \cdots, N_1$ will be important for us in the sequel, but the following property is valid for any n:

$$\eta_{n,1}(f) = 0 \quad \text{for any} \quad f \in \Pi_{N_0}(K_0). \tag{1}$$

Here and in what follows by $\Pi_N(A)$ we denote the set of functions coinciding on the set A with some polynomial of degree not greater than N, $\Pi_N := \Pi_N(\mathbb{R})$. To prove (1) for $n \leq N_0$ we can use the expansion $f = \sum_{j=0}^{N_0} \xi_{j,0}(f) e_{j,0}$ of f with respect to the basis $(e_{j,0})_{j=0}^{N_0}$ of Π_{N_0}. If $n > N_0$, then $\eta_{n,1} = \xi_{n,1}$ contains Π_{n-1} in its kernel.

Given f on K_0 let us denote by $Q_n(f, (x_k)_{k=1}^{n+1}, \cdot)$ (also by $Q_n(\cdot)$) the Newton interpolation polynomial of degree n for f with nodes at $x_1, \cdots, x_{n+1}$.

Let us consider the function $S_n(f,x) = Q_n(f, (x_k^{(0)})_{k=1}^{n+1}, x)$ for $n = 0, \cdots, N_0$ and

$$S_{N_0+r}(f,x) = Q_{N_0}(f, (x_k^{(0)})_{k=1}^{N_0+1}, x) + \sum_{k=M_1+1}^{M_1+r} \eta_{k,1}(f) e_{k,1}(x) \tag{2}$$

for $r = 1, \cdots, N_1 - M_1$. We see at once that $S_{N_0+r} \in \Pi_{N_0}(K_0 \setminus K_1)$ and $S_{N_0+r} \in \Pi_{max\{N_0,\, M_1+r\}}(K_1)$.

Lemma 1. *Given function f defined on K_0 and $n = 0, 1, \cdots, N_0 + N_1 - M_1$, the function $S_n(f, \cdot)$ interpolates f at the first $n+1$ points from the set* $\{x_1^{(0)}, \cdots, x_{N_0}^{(0)}, x_{M_1+1}^{(1)}, \cdots, x_{N_1+1}^{(1)}\}$.

Proof: The result is obvious for $n \le N_0$. Let $n = N_0 + r$ with $r = 1, \cdots, N_1 - M_1$. The polynomial Q_{N_0} interpolates f at the points $x_1^{(0)}, \cdots, x_{N_0}^{(0)}, x_{M_1+1}^{(1)}$. For $k \ge M_1 + 1$ the functions $e_{k,\,1}$ take zero value at these points. Therefore, $S_{N_0+r}(f, x_j^{(0)}) = f(x_j^{(0)}),\ j = 1, \cdots, N_0 + 1$ and we need to check only the condition

$$S_{N_0+r}(f, x_m^{(1)}) = f(x_m^{(1)}),\ \ m = M_1 + 2, \cdots, M_1 + r + 1.$$

Since $S_{N_0+r}(f, \cdot)$ is defined just by the values of f at the points $x_1^{(0)}, \cdots,$ $x_{M_1+r+1}^{(1)}$, there will be no changes if we replace f by the polynomial

$$Q_{N_0+r} = Q_{N_0}(f, (x_k^{(0)})_{k=1}^{N_0+1}, \cdot) + \sum_{j=M_1+1}^{M_1+r} \beta_j(f)\ \ \Omega_j(\cdot),$$

where $\beta_j(f) = [x_1^{(0)}, \cdots, x_{N_0}^{(0)}, x_{M_1+1}^{(1)}, \cdots, x_{j+1}^{(1)}]f,\ \ \Omega_j(x) = \tilde{e}_{N_0-M_1,\,0}(x)\, \tilde{e}_{j,\,1}(x)$.

Substituting this in (2) we get $Q_{N_0}(Q_{N_0+r}) = Q_{N_0}$ and also $\eta_{k,\,1}(Q_{N_0}) = 0$, by (1). The function Ω_j takes zero value at all points $x_k^{(0)}$, $k = 0, \cdots, N_0 + 1$ that define the divided differences $\xi_{k,\,0}$. Hence, $\xi_{k,\,0}(\Omega_j) = 0$ and $\eta_{k,\,1}(\Omega_j) = \xi_{k,\,1}(\Omega_j)$. It follows that

$$\sum_{k=M_1+1}^{M_1+r} \eta_{k,\,1}(Q_{N_0+r})\ e_{k,\,1}(x) = \sum_{j=M_1+1}^{M_1+r} \beta_j(f) \sum_{k=M_1+1}^{M_1+r} \xi_{k,\,1}(\Omega_j)\, e_{k,\,1}(x).$$

At the point $x = x_m^{(1)}$ the last sum equals $\sum_{k=j}^{m-1} \xi_{k,\,1}(\Omega_j)\, e_{k,\,1}(x_m^{(1)})$. The terms with $k < j$ disappear, since the points $x_1^{(1)}, \cdots, x_{k+1}^{(1)}$ are zeros of Ω_j for $j > k$. Also $e_{k,\,1}(x_m^{(1)}) = 0$ if $k \ge m$. Including all these zero terms we get the sum $\sum_{k=0}^{deg\,\Omega_j} \xi_{k,\,1}(\Omega_j)\, e_{k,\,1}(x_m^{(1)})$ which is the value of the interpolation polynomial for the function Ω_j at the point $x_m^{(1)}$ that is $\Omega_j(x_m^{(1)})$.

Therefore,

$$S_{N_0+r}(f, x_m^{(1)}) = Q_{N_0}(x_m^{(1)}) + \sum_{j=M_1+1}^{M_1+r} \beta_j(f)\ \ \Omega_j(x_m^{(1)}) = Q_{N_0+r}(x_m^{(1)}).$$

But the polynomial Q_{N_0+r} interpolates the function f at the point $x_m^{(1)}$. Thus, $S_{N_0+r}(f, x_m^{(1)}) = f(x_m^{(1)})$. □

The point of the lemma is that it allows one to interpolate functions locally. Suppose we have a chain of compact sets $K_0 \supset K_1 \supset \cdots \supset K_s \supset \cdots$ and finite systems of distinct points $(x_k^{(s)})_{k=1}^{N_s} \subset K_s$ for $s = 0, 1, \cdots$. Some part of the knots on K_{s+1} – let $(x_k^{(s+1)})_{k=1}^{M_{s+1}}$– belongs to the previous set $(x_k^{(s)})_{k=1}^{N_s}$. We will interpolate a given function f on K_s up to the degree N_s and then restrict the interpolation to the set K_{s+1}, where the degree of interpolation will be N_{s+1}, etc. As the diameters of K_s

decrease, the approximation properties of the interpolating polynomials will improve. Since the points of interpolation are chosen independently on functions, the approach allows us to construct topological bases in spaces of functions defined on rarefied sets.

3. **Estimations for fundamental Lagrange polynomials**

Given function f on a compact set $K \subset \mathbb{R}$, let $\omega(f, \cdot)$ be the modulus of continuity of f, that is $\omega(f,t) = sup\{\,|\,f(x) - f(y)\,| : x, y \in K,\ |x - y| \le t\},\ t > 0$. For $N \ge 1$ and distinct points $x_1, \cdots, x_{N+1} \in K$ with $x_1 < x_2 < \cdots < x_{N+1}$ let $e_{N+1}(x) = \prod_{k=1}^{N+1}(x - x_k),\ \xi_N(f) = [x_1, x_2, \cdots, x_{N+1}]f$ and $t = max_{k \le N}|\,x_{k+1} - x_k|$. Then from the representation

$$\xi_N(f) = \sum_{k=1}^{N+1} \frac{f(x_k)}{e'_{N+1}(x_k)} = \sum_{k=1}^{N}[f(x_k) - f(x_{k+1})] \cdot \sum_{j=1}^{k} \frac{1}{e'_{N+1}(x_j)}$$

we easily get

$$|\,\xi_N(f)\,| \le N^2\ \omega(f,t)\ (min_{k \le N}|\,e'_{N+1}(x_k)|\,)^{-1}. \tag{3}$$

All our subsequent considerations are related to Cantor-type sets. Let $\Lambda = (l_s)_{s=0}^{\infty}$ be a sequence such that $l_0 = 1$ and $0 < 2l_{s+1} \le l_s$ for $s \in \mathbb{N}_0 := \{0, 1, \cdots\}$. Let $K(\Lambda)$ be the Cantor set associated with the sequence Λ that is $K(\Lambda) = \bigcap_{s=0}^{\infty} E_s$, where $E_0 = I_{1,0} = [0,1]$, E_s is a union of 2^s closed *basic* intervals $I_{j,\,s}$ of length l_s and E_{s+1} is obtained by deleting the open concentric subinterval of length $h_s := l_s - 2l_{s+1}$ from each $I_{j,\,s}\ , j = 1, 2, ...2^s$. We will consider Cantor-type sets with the restriction only that

$$\exists A:\ \ l_s \le A \cdot h_s,\ \ \forall s. \tag{4}$$

Without loss of generality we suppose $A \ge 2$.

Let x be an endpoint of some basic interval. Then there exists the minimal number s (the *type* of x) such that x is the endpoint of some $I_{j,\,m}$ for every $m \ge s$.

By K_s we denote $K(\Lambda) \cap [0, l_s]$. Given K_s with $s \in \mathbb{N}_0$, let us choose the sequence $(x_n)_1^{\infty}$ by including all endpoints of basic intervals, using the rule of increase of the type. For the points of the same type we first take the endpoints of the largest gaps between the points of this type; here the intervals $(-\infty, x), (x, \infty)$ are considered as gaps. From points adjacent to the equal gaps, we choose the left one x and then $l_s - x$. Thus, $x_1 = 0, x_2 = l_s, x_3 = l_{s+1}, \cdots, x_7 = l_{s+1} - l_{s+2}, \cdots, x_{2^k+1} = l_{s+k}, \cdots$ (see [6] for more details).

Set

$$\mu_{s,\,N} := \frac{max_{x \in K_s}|\,e_N(x)|}{min_{j \le N}|\,e'_{N+1}(x_j)|}, \quad L_{N,\,j}(x) = \prod_{k=1,\,k \ne j}^{N} \frac{x - x_j}{x_k - x_j},$$

that is $L_{N,\,j}$ denotes the fundamental Lagrange polynomial.

Lemma 2. *Suppose the Cantor-type set $K(\Lambda)$ satisfies (4) and for $N \ge 1$ the points $(x_k)_1^{N+1} \subset K_s$ are chosen by the rule of increase of the type. Then*

$$\mu_{s,\,N} \le A^N \quad \textit{and} \quad max_{j \le N,\,x \in K_s}|\,L_{N,\,j}(x)| \le A^{N-1}.$$

Proof: For $N = 1$ the bounds are trivial. Let $N = 2^n + \nu$ with $0 \le \nu < 2^n$. Then $(x_k)_1^{N+1}$ consists of all endpoints of basic intervals of the type $s+n-1$ and $\nu+1$ points of the type $s+n$. Fix any $x \in K_s$ and $x_j,\ j \le N+1$.

By $(y_k)_1^N$ we denote the points $(x_k)_1^N$ arranged in the order of increase of distances $|\,x - x_k|$, that is $|\,x - y_k| = |\,x - x_{\sigma_k}| \uparrow$. Then $Y = (y_k)_1^N = \cup_{m=0}^n Y_{s+m}$ where $Y_q = \{y_k : \, h_q \le |\,x - y_k| \le l_q\}$, $q = s+n-1, \cdots, s$ and $Y_{s+n} = \{y_k : \, |\,x - y_k| \le l_{s+n}\}$.

Similarly, $Z = (z_k)_1^N$ consists of the points $x_k, 1 \le k \le N+1, k \ne j$ but here $|\,x_j - z_k| = |\,x_j - x_{\tau_k}| \uparrow$. As before, $Z = \cup_{m=0}^n Z_{s+m}$ with $Z_{s+n} = \{z_k : \, |\,x_j - z_k| = l_{s+n}\}$, $Z_q = \{z_k : \, h_q \le |\,x_j - z_k| \le l_q\}$ for $q = s+n-1, \cdots, s$. Let $a_q = |\,Y_q\,|$, $b_q = |\,Z_q\,|$ be the cardinalities of the corresponding sets. Since the points $(x_k)_1^{N+1}$ are uniformly distributed on K_s, it follows that the numbers of points x_k in two basic intervals $I_{i,\,q}, I_{j,\,q}$ of equal length are the same or differ by 1 (see [6] for details). But the point x_j is not included into the computation of b_q. Hence we have for $q = s+n, \cdots, s$ the following inequality

$$a_{s+n} + \cdots + a_q \ge b_{s+n} + \cdots + b_q. \tag{5}$$

Next, $|\,e_N(x)\,| = \prod_{k=1}^N |\,x - y_k| \le l_{s+n}^{a_{s+n}} \cdots l_s^{a_s}$, $|\,e'_{N+1}(x_j)| = \prod_{k=1}^N |\,x_j - z_k| \ge l_{s+n}^{b_{s+n}} \cdot h_{s+n-1}^{b_{s+n-1}} \cdots h_s^{b_s}$ and

$$\frac{|\,e_N(x)|}{|\,e'_{N+1}(x_k)|} \le \prod_{k=s}^{s+n} l_k^{a_k - b_k} \prod_{k=s}^{s+n-1} (l_k/h_k)^{b_k}. \tag{6}$$

Let us show that the first product in (6) does not exceed 1. Since $a_{s+n} \ge b_{s+n}$ and $l_{s+n} < l_{s+n-1}$, we get $\prod_{k=s+n-1}^{s+n} l_k^{a_k - b_k} \le l_{s+n-1}^{a_{s+n} + a_{s+n-1} - b_{s+n} - b_{s+n-1}}$. From (5) we see that the new degree of l_{s+n-1} is not negative. Therefore,

$$\prod_{k=s+n-2}^{s+n} l_k^{a_k - b_k} \le l_{s+n-2}^{\sum_{s+n-2}^{s+n} (a_k - b_k)}.$$

We continue in this fashion eventually obtaining $l_s^{\sum_s^{s+n} (a_k - b_k)} = 1$, as $\sum_s^{s+n} a_k = \sum_s^{s+n} b_k = N$.

The second product on the right in (6) can be estimated from above by $A^{\sum_s^{s+n} b_k} = A^N$. The points x and x_j were chosen in arbitrary way; thus we get the first desired inequality.

The polynomials $L_{N,\,j}$ can be handled in the same way; the only difference is that now $(y_k)_1^{N-1}$ and $(z_k)_1^{N-1}$ are the points $(x_k)_{k=1,\,k \ne j}^N$ properly rearranged. □

Example 1. Let $K(\Lambda)$ be the classical Cantor set. For any s the sequences $(\mu_{s,\,N})_N$ and $(max_{j \le N,\, x \in K_s} |\,L_{N,\,j}(x)|)_N$ have exponential growth.

Proof: Without loss of generality we can take $s = 0$. Let us define the Cantor sequence $(c_k)_1^\infty$ as $0, 1, 2, 3, 6, 7, 8, 9, 18, \cdots$. Here for $q = 1, 2, \cdots$ and $m = 1, \cdots, 2^q$ we get $c_{2^q+m} = 2 \cdot 3^{q-1} + c_m$. Fix $N = 2^n - 1$. The polynomial e_{N+1} has as its zeros all endpoints of the type $\le n-1$, that is the points $(3^{-n+1} c_k)_{k=1}^{2^n}$. In the same way as in [6] we take $d_1 = 0$, $d_2 = l_{n-1}, d_3 = l_{n-2} - l_{n-1}, \cdots, d_k = l_{n+1-k} - d_{k-1}$, so $d_k = l_{n+1-k} - l_{n+2-k} + \cdots + (-1)^k\, l_{n-1}$.

For the point $x_N = 1/3 - 1/9 + \cdots + (-1)^n \, 3^{-n+1}$ we have

$$|\, e'_{N+1}(x_N)| = \prod_{k=0}^{n-1} \prod_{j=2^k+1}^{2^{k+1}} (3^{-n+1}\, c_j - d_{k+1}).$$

On the other hand, $|\, e_N(l_n)| = |\, e_{N+1}(l_n)| \cdot |\, x_{N+1} - l_n|^{-1} \geq |\, e_{N+1}(l_n)|$ with $x_{N+1} = 1 - x_N$. Here $|\, e_{N+1}(l_n)| = l_n \prod_{k=0}^{n-1} \prod_{j=2^k+1}^{2^{k+1}} (3^{-n+1}\, c_j - l_n)$. Therefore,

$$\mu_{\,0,\,N} \geq \; l_n \;\; \frac{l_{n-1} - l_n}{l_{n-1}} \; \frac{2l_{n-1} - l_n}{l_{n-1}} \; \frac{l_{n-2} - l_n}{2l_{n-1}} \; \prod_{k=2}^{n-1} \prod_{j=2^k+1}^{2^{k+1}} \frac{3^{-n+1}\, c_j - l_n}{3^{-n+1}\, c_j - d_{k+1}}.$$

For $k \geq 2$ we get $d_{k+1} > \frac{2}{9}\, l_{n-k-1}$, as is easy to check. If $2^k + 1 \leq j \leq 2^{k+1}$, then $c_j \leq 3^k$ and $3^{-n+1}\, c_j \leq l_{n-k-1}$. Therefore all terms in the product $\prod_j$ exceed $6/5$. Thus,

$$\mu_{\,0,\,N} \geq l_n \cdot \frac{40}{27} \cdot \left(\frac{6}{5}\right)^{2^n - 4} > l_n \cdot \left(\frac{6}{5}\right)^{2^n - 2} = \frac{25}{36}\, exp\,[(N+1) \cdot ln\frac{6}{5} - \frac{ln3}{ln2} \cdot ln(N+1)].$$

The same arguments are valid for the lower bound of $|\, L_{N,\,N}(l_n)|$. □

The values of the fundamental Lagrange polynomials do not exceed 1 if the nodes of interpolation are chosen as Fekete points. In our case the system of nodes of interpolation is monotone, that is we choose the point x_{N+1} for the points $(x_k)_1^N$ already fixed. Thus, $(x_k)_1^\infty$ is a kind of Leja sequence.

For more rarefied Cantor type sets the bounds of Lemma 2 can be considerably improved. Let us consider for example the set $K^{(\alpha)}$, that is the Cantor set associated with the sequence $1, l_1, l_1^\alpha, \cdots, l_1^{\alpha^s}, \cdots$. Here $\alpha > 1$ and without loss of generality let $6l_1 \leq 1$.

Example 2. If $N_s\, l_s^{\alpha-1} \leq 1$, then for $1 \leq N \leq N_s$ we get $\mu_{s,\,N} \leq e^3$ and $max_{j \leq N,\, x \in K_s} |\, L_{N,\,j}(x)| \leq e^3$.

Proof: In this case the sequence l_s/h_s decreases, therefore the second product in (6) does not exceed $(l_s/h_s)^{N_s}$. Here $1 - 2l_s^{\alpha-1} \geq \frac{N_s - 2}{N_s}$. Hence, $(l_s/h_s)^{N_s} \leq (\frac{N_s}{N_s - 2})^{N_s} < e^3$ if $N_s \geq 6$. If $N_s \leq 5$, then $(l_s/h_s)^{N_s} \leq h_0^{-5} \leq (3/2)^5$, which is also smaller than e^3. □

Remark. By Lemma 1 in [6], the condition $N_s\, l_s^{\alpha-1} \leq 1$ implies that the first N_s Leja points are uniformly distributed on the set $K^{(\alpha)} \cap [0, l_s]$.

4. Interpolating bases

Fix $s \in \mathbb{N}$. Let natural numbers n_{s-1}, n_s be given with $n_{s-1} \leq n_s$. Set $N_s = 2^{n_s}$ and $N_{s-1} = 2^{n_{s-1}}$. Given N with $1 \leq N \leq N_{s-1}$ we choose the points $(x_k^{(s-1)})_{k=1}^{N_{s-1}+1}$ on K_{s-1} and $(x_k)_{k=1}^N$ on K_s by the rule of increase of the type. As above, $\xi_{k,\,s-1}(f) = [x_1^{(s-1)}, \cdots, x_{k+1}^{(s-1)}]f$, $e_{k,\,s-1}(x) = \prod_{j=1}^k (x - x_j^{(s-1)})|_{K_{s-1}}$ for $k = 1, 2, \cdots, N_{s-1}$. Also let $e_N(y) = \prod_{j=1}^N (y - x_j)|_{K_s}$

Lemma 3. *For fixed* $f \in C(K(\Lambda))$, $x \in K_s$ *let* $\tilde{\xi}(f) = [x_1, \cdots, x_N, x]f$, $\tilde{\eta}(f) = \tilde{\xi}(f) - \sum_{k=N}^{N_{s-1}} \tilde{\xi}(e_{k,\,s-1})\xi_{k,\,s-1}(f)$. *Then*

$$|\, \tilde{\eta}(f)\, e_N(x)\,| \leq N_{s-1}^4\, A^{2\,N_{s-1}}\, \omega(f, l_{s-1}).$$

In the case $K(\Lambda) = K^{(\alpha)}$ *we have* $|\,\tilde{\eta}(f)\,e_N(x)\,| \le e^6\,N_{s-1}^4\,\omega(f, l_{s-1})$, *provided the condition* $N_s\,l_s^{\alpha-1} \le 1$.

Proof: By $\tilde{e}$ we denote the function $\tilde{e}(y) = (y-x)\,e_N(y)$. Then by (3) $|\,\tilde{\xi}(f)\,| \le N^2\ \omega(f, l_s)\ (min_{j\le N}|\,\tilde{e}\,'(x_j)|\,)^{-1}$. Since $e_N(x)/\tilde{e}\,'(x_j) = -L_{N,j}(x)$, Lemma 2 now implies

$$|\,\tilde{\xi}(f)\,e_N(x)\,| \le N^2\,A^{N-1}\,\omega(f, l_s). \tag{7}$$

The representation $\tilde{\xi}(e_{k,\,s-1}) = e_{k,\,s-1}(x)/e_N(x) + \sum_{j=1}^N\ e_{k,\,s-1}(x_j)/\tilde{e}\,'(x_j)$ gives

$$|\,\tilde{\xi}(e_{k,\,s-1})\,\xi_{k,\,s-1}(f)\,e_N(x)\,| \le$$

$$|\,\xi_{k,\,s-1}(f)\,e_{k,\,s-1}(x)\,| + \sum_{j=1}^N \frac{|\,e_{k,\,s-1}(x_j)\,|}{|\,\tilde{e}\,'(x_j)\,|}\cdot\frac{|\,e_N(x)\,|}{min_{i\le k}|\,e'_{k+1,\,s-1}(x_i)|}\ \ k^2\,\omega(f, l_{s-1}).$$

The first term on the right does not exceed $k^2\,A^k\,\omega(f, l_{s-1})$, by (3) and Lemma 2. The parts of the two fractions in the second sum will be considered cross-wise. Applying Lemma 2 twice we get

$$|\,\tilde{\xi}(e_{k,\,s-1})\,\xi_{k,\,s-1}(f)\,e_N(x)\,| \le (1 + N\,A^{N-1})\,k^2\,A^k\,\omega(f, l_{s-1}).$$

Clearly, $\sum_1^n k^2 A^k \le 5/8\ A^n\,n^3$ for $n \ge 2$. Summing over k and taking into account (7), we get the general estimation of $|\,\tilde{\eta}(f)\,e_N(x)\,|$.

In the same manner we obtain the desired bound in the case $K(\Lambda) = K^{(\alpha)}$. □

The task is now to show that the biorthogonal system suggested in [5] as a basis for the space $\mathcal{E}(K(\Lambda))$ forms a topological basis in the space $C(K(\Lambda))$ as well, provided a suitable choice of degrees of polynomials.

Given a nondecreasing sequence of natural numbers $(n_s)_0^\infty$, let $N_s = 2^{n_s}$, $M_s^{(l)} = N_{s-1}/2+1$, $M_s^{(r)} = N_{s-1}/2$ for $s \ge 1$ and $M_0 = 1$. Here, (l) and (r) mean *left* and *right* respectively. For any basic interval $I_{j,s} = [a_{j,s}, b_{j,s}]$ we choose the sequence of points $(x_{n,j,\,s})_{n=1}^\infty$ using the rule of increase of the type.

As in [5] we take $e_{N,1,0} = \Pi_{n=1}^N(x - x_{n,1,0}) = \Pi_1^N(x - x_n)$ for $x \in K(\Lambda)$, $N = 0, 1, \cdots, N_0$. For $s \ge 1$, $j \le 2^s$ let $e_{N,j,\,s} = \Pi_{n=1}^N(x - x_{n,j,\,s})$ if $x \in K(\Lambda)\cap I_{j,s}$, and $e_{N,j,\,s} = 0$ on $K(\Lambda)$ otherwise. Here, $N = M_s^{(a)}, M_s^{(a)}+1, \cdots, N_s$ with $a = l$ for odd j and $a = r$ if j is even. The functionals are given as follows: for $s = 0, 1, \cdots$; $j = 1, 2, \cdots, 2^s$ and $N = 0, 1, \cdots$, let $\xi_{N,j,\,s}(f) = [x_{1,j,\,s}, \cdots, x_{N+1,j,\,s}]f$. Set $\eta_{N,1,0} = \xi_{N,1,0}$ for $N \le N_0$. Every basic interval $I_{j,s}$, $s \ge 1$, is a subinterval of a certain $I_{i,s-1}$ with $j = 2i-1$ or $j = 2i$. Let

$$\eta_{N,j,\,s}(f) = \xi_{N,j,\,s}(f) - \sum_{k=N}^{N_{s-1}} \xi_{N,j,\,s}(e_{k,\,i,\,s-1})\,\xi_{k,\,i,\,s-1}(f)$$

for $N = M_s^{(a)}, M_s^{(a)}+1, \cdots, N_s$. As before, $a = l$ if $j = 2i-1$, and $a = r$ if $j = 2i$.

The important difference from the previous case given in [5] is that the space $\mathcal{E}(K(\Lambda))$ is nuclear, therefore, by the Dynin-Mityagin theorem, any topological basis there is absolute. On the other hand in the case of the Banach spaces $C(K(\Lambda))$ any topological basis is not even unconditional. By [12] the space $C[0,1]$ has no unconditional basis. But by Miliutin [14] all spaces $C(K)$ are isomorphic between themselves for compact sets of cardinality of the continuum. Thus we have to enumerate the elements $(e_{N,\,j,\,s})_{s=0,\,j=1,\,N=M_s}^{\infty,\;\;2^s,\;\;N_s}$ in a reasonable way. We arrange them by increasing the level s. Elements of the same level are ordered by increasing the degree, that is with respect to N. For fixed s and N the elements $e_{N,\,j,\,s}$ are ordered by increasing j, that is from left to right. In this way we introduce a injective function $\sigma\,:\,(N,\,j,\,s)\mapsto M\in\mathbb{N}$. At the beginning we have for zero level: $\sigma(0,\,1,\,0)=1,\cdots,\sigma(N_0,\,1,\,0)=N_0+1$. Since the degree of the first element on $I_{1,1}$ is greater that on $I_{2,1}$, we start the first level from $e_{N_0/2,\,2,\,1}\,:\,\sigma(N_0/2,\,2,\,1)=N_0+2,\,\sigma(N_0/2+1,\,1,\,1)=N_0+3,\,\sigma(N_0/2+1,\,2,\,1)=N_0+4,\cdots,\sigma(N_1,\,2,\,1)=N_0+1+2(N_1-N_0/2)+1=2(N_1+1)$ and we finish all elements of the first level. For $s=2$ we have two elements $e_{N_1/2,\,2,\,2},\,e_{N_1/2,\,4,\,2}$ of the smaller degree, so they have a priority: $\sigma(N_1/2,\,2,\,2)=2(N_1+1)+1,\,\sigma(N_1/2,\,4,\,2)=2(N_1+1)+2$. Then $\sigma(N_1/2+1,\,1,\,2)=2(N_1+1)+3,\sigma(N_1/2+1,\,2,\,2)=2(N_1+1)+4,\cdots,\sigma(N_2,\,4,\,2)=2(N_1+1)+4(N_2-N_1/2)+2=4(N_2+1)$. Continuing in this manner after completing of the $s-$th level we get the value $\sigma(N_s\,,\,2^s,\,s)=2^s(N_s+1)$.

By injectivity of the function σ there exists the inverse function σ_{-1}. Let $f_m=e_{\sigma_{-1}(m)},\;m\in\mathbb{N}$.

Theorem 1. *Let a Cantor-type set $K(\Lambda)$ satisfy (4). Then for any bounded sequence $(N_s)_0^\infty$ the system $(f_m)_1^\infty$ forms a Schauder basis in the space $C(K(\Lambda))$.*

Proof: Given $f\in C(K(\Lambda))$ by $S_M(f,\cdot)$ we denote the $M-$th partial sum of the expansion of f with respect to the system $(f_m)_1^\infty$, that is $S_M(f,x)=\sum\eta_{N,j,\,s}(f)\,e_{N,j,\,s}(x)$, where the sum is taken over all $N,\,j,\,s$ with $\sigma(N,j,\,s)\le M$. If $1\le M\le N_0+1$, then $S_M(f,x)=Q_{M-1}(f,(x_{n,1,0})_{n=1}^M,x)$. The next function S_{N_0+2} is not a polynomial on $I_{1,0}$. The restriction of S_{N_0+2} to the interval $I_{1,1}$ is Q_{N_0}, whereas $S_{N_0+2}|_{I_{2,1}}=Q_{N_0}+\eta_{N_0/2,2,\,1}(f)\,e_{N_0/2,2,\,1}$. In both cases we get the polynomials of degree N_0 that interpolate f at $N_0/2+1$ points each. And always the subscript M gives the total number of points where S_M interpolates f.

Continuing in this way we see that the restriction of the function $S_{2^p(N_p+1)}$ to any interval $I_{j,\,p}\,,\;j=1,\cdots,2^p$, coincides with $Q_{N_p}(f,(x_{n,j,\,p})_{n=1}^{N_p+1},\cdot)$. Adding the next terms $\eta(f)\,e$ to $S_{2^p(N_p+1)}$ we get on the intervals $I_{j,\,p+1}\,,\;j=1,\cdots,2^{p+1}$ certain polynomials of degree N_p that interpolate f at some points. Increasing M we get $S_{2^{p+1}N_p+1}$ that has a degree N_p on $I_{1,\,p+1}$ and interpolates f on this interval at N_p+1 points; so here it is the usual interpolating polynomial. Adding the next $2^{p+1}-1$ terms to the sum $S_{2^{p+1}N_p+1}$ we get $S_{2^{p+1}(N_p+1)}$. The restriction of this function to any interval $I_{j,\,p+1},\;j=1,\cdots,2^{p+1}$ gives $Q_{N_p}(f,(x_{n,j,\,p+1})_{n=1}^{N_p+1},\cdot)$ and $S_{2^{p+1}(N+1)}|_{I_{j,\,p+1}}$ produces $Q_N(f,(x_{n,j,\,p+1})_{n=1}^{N+1},x)$ for $N\ge N_p$. It will continue up to the value $N=N_{p+1}$, after which we do the next splitting.

Suppose $\eta_{N,j,\,s}(f)=0$ for all $N,j,\,s$. Then, by considering step by step all triples $\sigma_{-1}(m),\;m\in\mathbb{N}$, we get $\xi_{N,j,\,s}(f)=0$ for all $N,\,j,\,s$. The set of nodes of the corresponding divided differences is dense in $K(\Lambda)$. Therefore, $f=0$ and the expansion $f=\sum\eta_{N,j,\,s}(f)\,e_{N,j,\,s}(x)$ is unique. We need to check only the convergence of $S_M(f,\cdot)$ to f in the norm of the space $C(K(\Lambda))$.

Let $N_s \le B$ for $s \in \mathbb{N}_0$. Fix $f \in C(K(\Lambda))$, $\varepsilon > 0$ and s_ε such that $\omega(f, l_{s_\varepsilon}) \le B^{-4} A^{-2B} \varepsilon$. Let $M_\varepsilon = 2^{s_\varepsilon}(N_{s_\varepsilon} + 1)$. For any $M \ge M_\varepsilon$ we get $2^{s-1}(N_{s-1} + 1) \le M < 2^s(N_s + 1)$ with $s \ge s_\varepsilon + 1$.

Fix $x \in K(\Lambda)$. Without loss of generality let $x \in K(\Lambda) \cap [0, l_s]$.

If $2^{s-1}(N_{s-1} + 1) \le M \le 2^s N_{s-1}$, then

$$S_M(f, x) = Q_{N_{s-1}}(f, (x_{n,1,s-1})_{n=1}^{N_{s-1}+1}, x) + \sum \eta_{N,1,s}(f)\, e_{N,1,s}(x), \tag{8}$$

where the sum is taken over all N, j, s with $2^{s-1}(N_{s-1} + 1) < \sigma(N, j, s) \le M$. The degree N_{s-1} will appear for the first time when $\sigma_{-1}(2^s N_{s-1} + 1) = (N_{s-1}, 1, s)$. Thus for the values N in (8) we have $N \le N_{s-1} - 1$.

For the second case $2^s N_{s-1} + 1 \le M < 2^s(N_s + 1)$ we get

$$S_M(f, x) = Q_N(f, (x_{n,1,s})_{n=1}^{N+1}, x)$$

with some N, $N_{s-1} \le N \le N_s$.

Let us consider at the beginning the simpler second case. With the notation $\tilde{\xi}(f) = [x_{1,1,s}, \cdots, x_{N+1,1,s}, x]f$, we have the polynomial $\tilde{Q}_{N+1}(\cdot) = Q_N(\cdot) + \tilde{\xi}(f)\, e_{N+1,1,s}(\cdot)$ that interpolates f also at the point x. Therefore here $f(x) - S_M(f, x) = \tilde{\xi}(f)\, e_{N+1,1,s}(x)$ and as in (7)

$$|\, \tilde{\xi}(f)\, e_{N+1}(x)\, | \le (N + 1)^2 A^N \omega(f, l_s) \le (B + 1)^2 A^B \omega(f, l_s),$$

which does not exceed ε.

For the case $2^{s-1}(N_{s-1} + 1) \le M \le 2^s N_{s-1}$, we denote the last term of the sum in (8) by $\eta_{R,1,s}(f)\, e_{R,1,s}(x)$. As it was remarked before, $R \le N_{s-1} - 1$. We can use Lemma 3 with $N = R + 1$. Here $\tilde{\xi}(f) = [x_{1,1,s}, \cdots, x_{R+1,1,s}, x]f$ and $\tilde{\eta}(f) = \tilde{\xi}(f) - \sum_{k=R+1}^{N_{s-1}} \tilde{\xi}(e_{k,1,s-1}) \xi_{k,1,s-1}(f)$. Then by Lemma 1 the function $S_M(f, \cdot) + \tilde{\eta}(f)\, e_{R+1,1,s}(\cdot)$ interpolates f at the point x. Therefore, $|\, f(x) - S_M(f, x)\, | = |\, \tilde{\eta}(f)\, e_{R+1,1,s}(x)\, | \le \varepsilon$ by Lemma 3 and because of the choice of s_ε. Therefore, $|\, f(x) - S_M(f, x)\, | \le \varepsilon$ for any $M \ge M_\varepsilon$, which is the desired conclusion.

□

In the case $K = K^{(\alpha)}$ one can choose an unbounded sequence (N_s) such that the series $\sum \eta_{N,j,s}(\cdot)\, e_{N,j,s}$ will approximate functions from the Hölder class (that is functions with $\omega(f, t) \le C\, t^\delta$ for some C and δ) in the norm of $C(K)$. For example, the condition $N_s\, l_s^{min\{\delta/4, \alpha-1\}} \to 0$, as $s \to \infty$ provides both the convergence of the series to f and the applicability of Lemma 3.

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